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Week 11

(11) 3x3 1 block

1x1 3 block $\xleftrightarrow[\text{size}]{\text{max block}}$ $m(s) = (s-1)^3 \leftarrow$ $m_1 = 3$ algebric multiplicity

$d(s) = (s-1)^6 \leftarrow$ $r_1 = 6$ geometric multiplicity

in total 4 block

$\updownarrow$ in total 4 blocks

$\dim(N(A_1^1)) = 4 \xrightarrow{+1 \text{ change}} \dim(N(A_1^2)) = 5 \xrightarrow{+1 \text{ change}} \dim(N(A_1^3)) = 6$

number of blocks having size ≥ 2 is $5-4=1$

number block having size: 3 is $6-5=1$

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}$$

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Ex: A is a 7×7 matrix with $\lambda_1=5$ $l_1=4$, $\lambda_2=2$ $l_2=3$ where λ_1 and λ_2 are eigenvalues and l_1 and l_2 are their geometric multiplicities. How many jordan form A ~~have~~ might ~~might~~ have?

1

$d(s)=(s-5)^4(s-2)^3$ $\rightarrow$ That means we have a 4×4 matrix wit 4 $\lambda_1=5$ eigenvalues and a 3×3 matrix with 3 $\lambda_2=2$ eigen values

Hence lets draw a 4×4 matrix and 3×3 matrix and find all jordan forms

For 4×4 matrix

(a) $m(s)=(s-5)^4$ $d(s)=(s-5)^4$ $\dim(N(A_1^1))=1$, $\dim(N(A_1^2))=2$, $\dim(N(A_1^3))=3$
$\dim(N(A_1^4))=4$

A single 4×4 block

$$\tilde{A}=\begin{bmatrix}5&0&0&0\\1&5&0&0\\0&1&5&0\\0&0&1&5\end{bmatrix}$$

(b) $m(s)=(s-5)^3$ $d(s)=(s-5)^4$ $\dim(N(A_1^1))=2$, $\dim(N(A_1^2))=3$
$\dim(N(A_1^3))=4$

$$\tilde{A}=\begin{bmatrix}5&0&0&0\\1&5&0&0\\0&1&5&0\\0&0&0&5\end{bmatrix} \rightarrow$$

A single 3×3 block
A single 1×1 block

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(c) $m(s)=(s-1)^2$, $d(s)=(s-1)^4$

$\dim(N(A_1^1))=2$, $\dim(N(A_1^2))=4$

$$\tilde{A}=\left[\begin{array}{cc|cc} s & 0 & 0 & 0 \\ 1 & s & 0 & 0 \\ \hline 0 & 0 & s & 0 \\ 0 & 0 & 1 & s \end{array}\right]$$

$\rightarrow$ 2 2×2 block

(d) $m(s)=(s-1)^2$, $d(s)=(s-1)^4$

$\dim(N(A_1^1))=3$, $\dim(N(A_1^2))=4$

$$\tilde{A}=\left[\begin{array}{cc|c|c} s & 0 & 0 & 0 \\ 1 & s & 0 & 0 \\ \hline 0 & 0 & s & 0 \\ \hline 0 & 0 & 0 & s \end{array}\right]$$

$\rightarrow$ 1 2×2 block
2 1×1 block

(e) $m(s)=(s-1)^1$ $d(s)=(s-1)^4$

$$\tilde{A}=\left[\begin{array}{c|c|c|c} s & 0 & 0 & 0 \\ \hline 0 & s & 0 & 0 \\ \hline 0 & 0 & s & 0 \\ \hline 0 & 0 & 0 & s \end{array}\right]$$

$\rightarrow$ 4 1×1 block

So for a 4×4 matrix there are 5 possibilities

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3x3 matrix

(a) $m(s) = (s-2)^3 \quad d(s) = (s-2)^3$

$\dim(N(A_1^1)) = 1, \quad \dim(N(A_1^2)) = 2, \quad \dim(N(A_1^3)) = 3$

$$\tilde{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \text{A single } 3\times3 \text{ block}$$

(b) $m(s) = (s-2)^2 \quad d(s) = (s-2)^3$

$\dim(N(A_1^1)) = 2, \quad \dim(N(A_1^2)) = 3$

$$\tilde{A} = \left[\begin{array}{cc:c} 2 & 0 & 0 \\ 1 & 2 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right] \rightarrow \begin{array}{l} 1 \quad 2\times2 \text{ block} \\ 1 \quad 1\times1 \text{ block} \end{array}$$

(c) $m(s) = (s-2)^1 \quad d(s) = (s-2)^3$

$\dim(N(A_1^1)) = 3$

$$\tilde{A} = \left[\begin{array}{c:c:c} 2 & 0 & 0 \\ \hdashline 0 & 2 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right] \rightarrow 3 \quad 1\times1 \text{ block}$$

So for a 3x3 matrix there are 3 possibilities

Hence for a 7×7 matrix with $\lambda_1 = 5$ $m_1 = 4$ $\lambda_2 = 2$ $m_2 = 3$ there are 15 possibilities

$15 = 5 \times 3$

total possibilities for jordan forms of 7×7 matrix where $\lambda_1 = 5$, $m_1 = 4$ $\lambda_2 = 2$, $m_2 = 3$

possibilities for a 4×4 matrix

possibilities for a 3×3 matrix

Ex.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5}$$

(a) find $d(s)$

A is a 5×5 matrix

$$d(s) = \det(sI - A) = \det\left(\begin{bmatrix} s-1 & 0 & 0 & 0 & 0 \\ -5 & s-1 & 0 & 0 & 0 \\ 0 & 0 & s-1 & 0 & 0 \\ 0 & 0 & -5 & s-1 & 0 \\ 0 & 0 & 0 & 0 & s-1 \end{bmatrix} \right) = (s-1)^5$$

$\lambda_1 = 1$

$r_1 = 5$

(b) Find $m(s)$, m_1?

$$A_1 = A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find $N(A_1) \Rightarrow A_1 x = \theta$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$

$x_3 = 0$

x_2, x_4, x_5 arbitrary

Hence

$$N(A_1) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\dim N(A_1) = 3$

<u>Find $N(A_1^2)$</u>

$$A_1^2 = A_1 A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $N(A_1^2)$ $\quad A_1^2 x = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

x_1, x_2, x_3, x_4, x_5 arbitrary

$$N(A_1^2) = \text{Span}\left\{ \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{span}(N(A_1))}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\dim(N(A_1^2)) = 5 \rightarrow r_1 \rightarrow$ geometric multiplicity

hence $m_1 = 2 \rightarrow$ algebraic multiplicity

Summary

$d(s) = (s-1)^5$

$m(s) = (s-1)^2$

$$\dim(N(A_1)) = 3 \xrightarrow{+2 \text{ change}} , \quad \dim(N(A_1^2)) = 5$$

$\dim(N(A_1)) = 3 \downarrow$ 3 jordan blocks

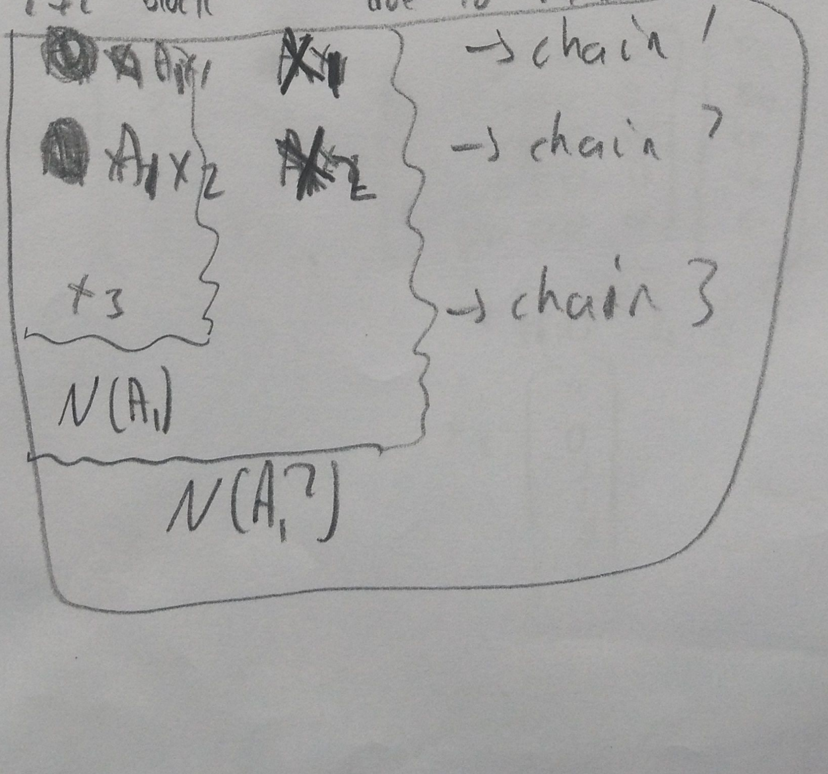
$\dim(N(A_1^2)) = 5 \downarrow$ 2 block have size 2×2 due to 2×2 chang

Hence

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

* That means there are 3 chains (3 blocks)

1st chains 2 vectors, due to 2×2 block, 2nd chain 2 vectors due to 2×2 block, 3rd chain 1 vector due to 1×1 block



$$k_1 = A_1 x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_a \\ k_b \\ k_c \\ k_d \\ k_e \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$5k_a = 1 \quad 5k_c = 0$

$k_a = \frac{1}{5} \quad k_c = 0 \quad k_b, k_d, k_e$ arbitrary

$$\underbrace{A_1 x_1 = k_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\in N(A_1^1)} ; \quad \underbrace{x_1 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\in N(A_1^2)}$$

first chain

$A_1 x_1, \ x_1$

$$k_2 = A_1 x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_a \\ k_b \\ k_c \\ k_d \\ k_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$k_a = 0 \quad k_c = \frac{1}{5} \quad k_b, k_d, k_e$ arbitrary

$$x_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5} \\ 0 \\ 1 \end{bmatrix} \qquad A_1 x_2, \ x_2$$

Let's find a vector x_3 such that

$\text{span}\{A_1 x_1, A_1 x_2, x_3\} = N(A_1)$

$\text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, x_3 \right\} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Let's take $x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \rightarrow$ possible

$\cancel{x_3}$

$P = x_1 \vdots A_1 x_1 \vdots x_2 \vdots A_1 x_2 \vdots x_3$

$$P = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & -2.5 & 0 & 5 \end{bmatrix}$$

$$\tilde{A} = P^{-1} A P \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Jordan form}$$